

# Symmetrized quartic polynomial oscillators and their partial exact solvability

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## Abstract

Sextic polynomial oscillator is probably the best known quantum system which is partially exactly *alias* quasi-exactly solvable (QES), i.e., which possesses closed-form, elementary-function bound states  $\psi(x)$  at certain couplings and energies. In contrast, the apparently simpler and phenomenologically more important quartic polynomial oscillator is *not* QES. A resolution of the paradox is proposed: The one-dimensional Schrödinger equation is shown QES after the analyticity-violating symmetrization  $V(x) = A|x| + Bx^2 + C|x|^3 + x^4$  of the quartic polynomial potential.

## Keywords:

quantum bound states; non-numerical methods; piecewise analytic potentials; quartic oscillators; quasi-exact states;

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03.65.Ge Solutions of wave equations: bound states

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# 1 Introduction

A (certainly, not entirely complete) description of the dramatic recent history of the discovery of existence of certain anomalous, elementary, harmonic-oscillator-resembling exceptional non-numerical bound-state solutions  $\psi(x)$  of certain non-harmonic-oscillator Schrödinger equations may be found described in the dedicated monograph by Ushveridze [1]. Although the author himself calls these models quasi-exactly solvable (QES), he immediately admits that the other authors may endow the same name with a different meaning (this is a rather philosophical subtlety, a deeper discussion of which may be found postponed to Appendix A). At the same time, Ushveridze claims that irrespectively of the rigorous definition of the QES concept, the popular quartic-oscillator potentials

$$V(x) = V^{(QO)}(x) = Ax + Bx^2 + Cx^3 + x^4. \quad (1)$$

*cannot* be considered QES.

The resulting absence of any non-numerical bound-state solutions of one-dimensional Schrödinger equations

$$-\frac{d^2}{dx^2}\psi_n(x) + V(x)\psi_n(x) = E_n\psi_n(x), \quad \psi_n(x) \in L^2(\mathbb{R}) \quad (2)$$

with interactions (1) is to be perceived as strongly unpleasant. Indeed, the simplified quantum model (1) + (2) plays the phenomenologically extremely useful role in quantum chemistry [2]. In parallel, it often serves as a mathematical and methodical laboratory in relativistic quantum field theory. *Pars pro toto*, let us just mention that at certain couplings, Eq. (1) defines the radial part of the famous Mexican-hat potential which samples the emergence of the Nambu-Goldstone bosons in systems with spontaneously broken symmetries (cf. picture Nr. 7 in [3]).

Secondly, one must mention the context of atomic, molecular and nuclear physics in which the double-well shape of the one-dimensional version of the Mexican-hat potential (using, say, a large negative  $B \ll 0$  in (1)) proves useful as leading to an apparent degeneracy of bound states with opposite parity. Unfortunately, this is related to the quantum tunneling phenomenon which is rather difficult to describe by non-numerical or even semi-numerical means as sampled, e.g., by the conventional perturbation theory (cf. also Refs. [4, 5, 6] in this context).

It seems worth adding that the Ushveridze's declaration of the non-QES status of the quartic-oscillator potentials (1) is only valid under certain tacit assumptions including, e.g., the conventional postulate that the potential should be analytic but, at the same time, that the coordinate  $x$  itself must be observable, i.e., that it must lie strictly on the real line. After one abandons one of the latter two assumptions, the Ushveridze's "no-go" theorem is not valid anymore. In the literature one can find several counterexamples defined along a suitable complex curve of  $x$  for which the corresponding quartic oscillator becomes QES (cf., e.g., [7, 8]).

The discovery of the latter counterexamples was preceded by the mathematical clarification of the consistent physical meaning of having complex  $x \in \mathbb{C}$  (cf. [9] or [10]). Such an extension of the scope of the quantum model building also inspired several phenomenologically oriented further

developments [11]. Nevertheless, we are not going to move along this line here, mainly because the loss of the reality of the coordinate changes the physics too much. In particular, the change implies such a growth of extent of necessary additional mathematics [12] that the underlying QES philosophy of having a “solvable” model gets, to a great extent, lost [13].

This is the reason why we shall keep the coordinate  $x$  real and remove, instead, the other Ushveridze’s tacit assumption of the analyticity of the potential. In such a setting we shall reveal and illustrate, constructively, the QES property of Schrödinger Eq. (2) with another quartic-polynomial-interaction potential

$$V(x) = A|x| + Bx^2 + C|x|^3 + x^4 \quad (3)$$

which is made spatially symmetric “by brute force”, i.e., at the expense of the loss of its analyticity in the origin.

## 2 Non-analytic QES models

In the context of mathematical physics polynomials (1) often play the role of benchmark interactions, say, in the tests of perturbation expansions [2]. For this reason, as we already indicated, it is not too fortunate that the related Schrödinger Eq. (2) must be solved, at *any* (real) triplet of coupling constants  $A$ ,  $B$  and  $C$ , by its brute-force numerical integration.

### 2.1 Inspiration: analytic sextic QES oscillators

In the light of the above comment it sounds like a paradox that elementary bound-state solutions do exist for the next, higher-degree polynomial potentials

$$V_{(sextic)}(x) = Ax^2 + Bx^4 + x^6, \quad (4)$$

at some exceptional, *ad hoc* couplings and energies at least [14, 15]. At the same time, in comparison with quartic oscillators (1), both the practical phenomenological implementations and/or the methodical implications of the sextic oscillators seem much less useful or relevant.

Let us add that the incomplete elementary solvability as exemplified by the sextic polynomial interaction (4) is characterized by the observation that a few exceptional bound states may still possess an elementary, harmonic-oscillator-resembling product form

$$\psi_n(x) = P(x) \exp W(x) \quad (5)$$

where  $P(x) = P(x, n)$  and  $W(x)$  are suitable polynomials.

### 2.2 Symmetrized WKB asymptotics

In what follows we intend to favor, in a somewhat complementary manner, the class of asymptotically fourth-power interactions (3). Our guiding idea was that in contrast to the partially solvable

model (4) (and also in a way reflecting the complicated Heun-function solvability of quartic oscillators as mentioned in Appendix A below), the fully analytic polynomial interaction potentials (1) defined on the whole real line are spatially asymmetric in general. This fact turned our attention to the possible Hermitian modifications of the quartic potentials and, in particular, to their spatially symmetrized version (3).

The main specific merit of potentials (3) is that they remain Hermitian and asymptotically analytic and confining, with the analyticity merely violated in the origin. On this background one only has to treat the point  $x = 0$  as a regular coordinate at which the logarithmic derivatives of the bound-state wave functions  $\psi_n(x)$  must be matched in standard manner as known and tested in the square-well models.

In what follows our main task will lie in showing that and how the entirely standard wave-function matching technique could be implemented in the QES context. Our main conclusion will be that the harmonic-oscillator-resembling elementary form of the quantum bound states may fairly easily survive the loss of the analyticity of the potential in the origin.

Polynomial  $W(x)$  in the standard QES ansatz (5) remains WKB-related and quantum-number-independent so that its form must be in a one-to-one correspondence with the asymptotic behavior of the potential. Naturally, there exists an immediate connection between  $W(x)$  and the WKB asymptotics of wave functions  $\psi_n(x)$ . Nevertheless, only a polynomial form of  $W(x)$  proves useful in the QES context.

Its specification is routine. In particular, one may easily construct polynomial  $W$  for the above-mentioned analytic sextic oscillators (4) yielding the proper normalizability-guaranteeing polynomial  $W_{(sextic)}(x) = -x^4/4 + \mathcal{O}(x^3)$  (cf. [14]). In contrast, one must be more careful in the case of our present non-analytic quartic oscillators (3). Branched, *non-analytic* asymptotics of any candidate for a QES wave function are obtained,

$$W_{(non-analytic)}(x) = \begin{cases} +x^3/3 + \mathcal{O}(x^2), & x \ll 0, \\ -x^3/3 + \mathcal{O}(x^2), & x \gg 0. \end{cases} \quad (6)$$

### 2.3 Wave-function matching in the origin

In a search for elementary quartic-oscillator bound states we must *necessarily* replace the analytic-function ansatz (5) by its suitable weaker version. Once we start, say, from the two-branched asymptotics (6) we may postulate, e.g.,

$$W_{(non-analytic)}(x) = \begin{cases} W_{(left)}(x) = +x^3/3 + ax^2 + bx, & x < 0, \\ W_{(right)}(x) = -x^3/3 + \tilde{a}x^2 - \tilde{b}x, & x > 0. \end{cases} \quad (7)$$

The related unavoidable loss of the analyticity of wave functions in the origin implies that we will have to treat our model (3) in a way similar to square wells, i.e., via matching the logarithmic derivatives of  $\psi_n(x)$  at the point of non-analyticity, i.e., at  $x = 0$ .

In the QES cases, in particular, we will have to assume that the conventional ansatz (5) gets split into two branches,

$$\psi_n^{(QES)}(x) = \begin{cases} P_{(left)}(x, n) \exp W_{(left)}(x), & x < 0, \\ P_{(right)}(x, n) \exp W_{(right)}(x), & x > 0, \end{cases} \quad (8)$$

where the form and construction of the two polynomials  $P_{(left/right)}(x, n)$  is still to be specified.

In general case, we may very easily replace, in addition, also the analytic polynomial interactions (1) by a broader class of their generalizations which would be also manifestly non-analytic in the origin and which could be rewritten in a six-parametric two-branched form

$$V_{(general)}(x) = \begin{cases} V_{(general \ left)}(x) = Ax + Bx^2 + Cx^3 + x^4, & x < 0, \\ V_{(general \ right)}(x) = \tilde{A}x + \tilde{B}x^2 + \tilde{C}x^3 + x^4, & x > 0. \end{cases} \quad (9)$$

This potential merely exhibits the left-right symmetry in asymptotic domain so that we shall rather limit our attention to the spatially symmetrized three-parametric potentials (3) re-written in a slightly modified, piecewise-analytic notation,

$$V^{(QES)}(x) = \begin{cases} qx + rx^2 + sx^3 + x^4, & x < 0, \\ -qx + rx^2 - sx^3 + x^4, & x > 0. \end{cases} \quad (10)$$

The well known consequence of the spatial symmetry of our Hamiltonians and potentials (10) is that the wave functions themselves must also be either symmetric or antisymmetric,  $\psi_n^{(QES)}(x) = \psi_n^{(even/odd)}(x)$ . Thus, in the QES context the key technical point is that it will be sufficient to stay, say, just on the left half-axis with  $x < 0$ . The resulting reduced QES ansatz will then read

$$\psi_n^{(even/odd)}(x) = \left( v_0^{(even/odd)} + v_1^{(even/odd)}x + \dots + v_N^{(even/odd)}x^N \right) \exp W_{(left)}(x), \quad x < 0. \quad (11)$$

Once we recall the necessary asymptotic boundary condition we get

$$s = s(a) = 4a = 4\tilde{a}, \quad r = r(a, b) = 4a^2 + 2b = 4\tilde{a}^2 + 2\tilde{b}. \quad (12)$$

This means that we have to put  $\tilde{a} = a$  and  $\tilde{b} = b$  in exponents (7).

The second requirement is the continuity of the wave function in the origin. This means that we must put  $P_{(left)}(0, n) = P_{(right)}(0, n)$ , i.e.,  $v_0^{(odd)} = 0$  and, say,  $v_0^{(even)} = 1$  (= the choice of normalization). In parallel, the continuity of the first derivative of the wave function in the origin leads to the other constraint, viz., to the specification of  $v_1^{(even)} = -b$  or, say,  $v_0^{(odd)} = 1$  (= the choice of normalization).

### 3 QES constructions

As long as the even-parity and odd-parity constructions remain entirely analogous, the explicit constructive description of the even solutions will be fully sufficient for our present illustrative purposes. In a preparatory step let us restrict attention to a small  $N = 2$ .

### 3.1 Even states $\psi_n^{(even)}(x)$ at $N = 2$

Once we choose  $N = 2$  in our QES ansatz (11) and denote  $P_{(left)}(x, n) = (1 + ux + vx^2)$ , we may evaluate the second derivative

$$\begin{aligned} \psi''(x) = e^{1/3 x^3 + ax^2 + bx} & \left[ x^6 v + (4av + u) x^5 + (2bv + 1 + 4a^2 v + 4au) x^4 + \right. \\ & + (4abv + 2bu + 4a + 4a^2 u + 6v) x^3 + (10av + 4u + 4abu + b^2 v + 4a^2 + 2b) x^2 + \\ & \left. + (4bv + 2 + b^2 u + 4ab + 6au) x + 2a + 2bu + b^2 + 2v \right]. \end{aligned} \quad (13)$$

In the light of Schrödinger equation this must be equal to expression  $[V(x) - E]\psi(x)$  where we may put  $E = -p$  and evaluate

$$\begin{aligned} [V(x) + p]\psi(x) = e^{1/3 x^3 + ax^2 + bx} & \left[ x^6 v + (u + vs) x^5 + (1 + us + vr) x^4 + \right. \\ & \left. + (s + ur + vq) x^3 + (r + uq + vp) x^2 + (q + up) x + p \right]. \end{aligned} \quad (14)$$

The respective individual coefficients at powers  $x^5, x^4, \dots, x^0$  must equal each other. This comparison generates the set of six algebraic equations

$$\begin{aligned} -vs + 4av &= 0, \quad 4a^2 v + 4au - vr - us + 2bv = 0, \\ 4a - s + 2bu + 4a^2 u + 4abv - ur - vq + 6v &= 0, \\ -r + 4abu + 4u + b^2 v + 4a^2 - vp + 10av + 2b - uq &= 0, \\ 4bv - q + 4ab + b^2 u + 2 + 6au - up &= 0, \quad 2a + 2bu + 2v - p + b^2 = 0. \end{aligned} \quad (15)$$

Due to relations (12) the first two items are just identities while the third one fixes the value of the last coupling constant,

$$q = q(a, b) = 4ab + 6. \quad (16)$$

The value of the energy becomes determined by the fourth equation,

$$E = E(a, b, v) = \frac{2u}{v} - 10a - b^2. \quad (17)$$

Once we construct just the even-parity state with property  $\psi'(0) = 0$ , i.e.,  $u = -b$ , we are left with the last two algebraic equations. The first one offers the two eligible wave-function-coefficient roots

$$v = v_{\pm} = \frac{1}{4b} \left( -2ab + 2 \pm 2\sqrt{a^2 b^2 - 2ab + 1 - 2b^3} \right) \quad (18)$$

while the last one yields

$$a = a_{\pm} = \frac{1}{20b} \left( -7b^3 - 8 \pm 3\sqrt{b^6 - 12b^3 + 16} \right) \quad (19)$$

and leaves just the real value of  $b \neq 0$  independently variable.

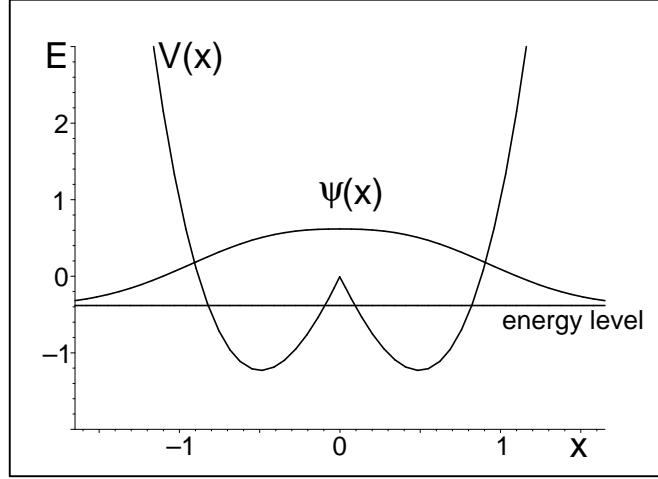


Figure 1: The QES ground state  $\psi_0(x)$  in the  $N = 2$  quartic potential (10) at positive  $b = 1$ . The effect of spike in  $V(x)$  proves negligible.

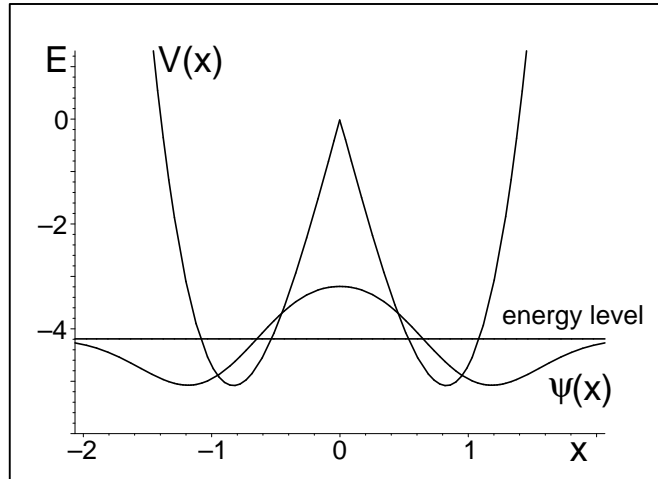


Figure 2: The second excited QES bound state  $\psi_2(x)$  in the  $N = 2$  quartic potential (10) at negative  $b = -1$ . The pronounced spike separates the potential, in effect, into two almost decoupled wells.

The resulting shapes of the QES potential (10) as well as of the related exact wave function  $\psi_n(x)$  are illustrated in Figures 1 and 2 where we choose  $v = v_+$  and  $a = a_+$ . We picked up two sample values of  $b = \pm 1$  and obtained the ground QES state and the second excited QES state, i.e., solutions with  $n = 0$  and  $n = 2$ , respectively.

We may conclude that although we sacrificed the analyticity of the wave functions in the origin, the resulting potential is defined by closed formula. At  $b = 1$  and  $x < 0$ , for example, we have

$$V(x) = 3 \left(1 + 1/\sqrt{5}\right) x + \left(2 + 9 \left(-1 + 1/\sqrt{5}\right)^2 / 4\right) x^2 + 3 \left(-1 + 1/\sqrt{5}\right) x^3 + x^4 \quad (20)$$

i.e., after numerical evaluation and symmetrization,

$$V_{(QES)}(x) = \begin{cases} 4.3416 x + 2.6875 x^2 - 1.6584 x^3 + x^4, & x < 0, \\ -4.3416 x + 2.6875 x^2 + 1.6584 x^3 + x^4, & x > 0. \end{cases} \quad (21)$$

We can summarize that the reconstruction of the  $b$ -parametrized family of the  $N = 2$  QES potentials is feasible and friendly. Numerically as well as non-numerically one can check the presence and size of the spike in the origin or determine the position of the local minima of the potential, etc. What remains to be added is the discussion of the general case using any preselected integer  $N$ .

### 3.2 General case

Once we assume that  $V(0) = 0$  and that  $x < 0$  we may abbreviate

$$V(x) - E = p + qx + rx^2 + sx^3 + x^4 \quad (22)$$

and differentiate our general polynomial Ansatz

$$\psi(x) = e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N v_k x^k \quad (23)$$

with  $v_N \neq 0$  (and with the entirely formal definitions of  $v_{-1} = v_{N+1} = v_{N+2} = 0$ ) yielding

$$\psi'(x) = (x^2 + 2ax + b)e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N v_k x^k + e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N (k+1) v_{k+1} x^k \quad (24)$$

and

$$\begin{aligned} \psi''(x) = & [(x^2 + 2ax + b)^2 + 2x + 2a] e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N v_k x^k + \\ & + 2(x^2 + 2ax + b)e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N (k+1) v_{k+1} x^k + e^{1/3 x^3 + ax^2 + bx} \sum_{k=0}^N (k+1)(k+2) v_{k+2} x^k. \end{aligned} \quad (25)$$

The latter expression may be inserted in Schrödinger equation  $\psi''(x) = (V(x) - E)\psi(x)$ . In the resulting relation between polynomials we already know that the contribution of the dominant power  $x^{N+3}$  fixes  $s = s(a) = 4a$ , demonstrating the mutual WKB-based large-coordinate



correspondence between the subdominant coupling and the subdominant exponent in the wave function. Similarly we are also aware of the triviality of the coefficient at the subdominant power  $x^{N+2}$  yielding  $r = r(a, b) = 4a^2 + 2b$ . What is new is the consequence of the vanishing of the coefficient at the sub-subdominant power  $x^{N+1}$  which parametrizes also our last QES coupling constant in an  $N$ -dependent manner,  $q = q(a, b, N) = 4ab + 2N + 2$ .

We are left with the set of  $N + 1$  recurrences for  $N + 1$  unknown coefficients  $v_k$ . This set is most easily presented as the formal eigenvalue problem

$$\begin{pmatrix} \mathcal{M}_{00} & \mathcal{M}_{01} & \mathcal{M}_{02} & 0 & \dots & 0 \\ \mathcal{M}_{10} & \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \ddots & \vdots \\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \mathcal{M}_{N-2N-1} & \mathcal{M}_{N-2N} \\ \vdots & \ddots & 0 & \mathcal{M}_{N-1N-2} & \mathcal{M}_{N-1N-1} & \mathcal{M}_{N-1N} \\ 0 & \dots & 0 & 0 & \mathcal{M}_{NN-1} & \mathcal{M}_{NN} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = p \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \quad (26)$$

where the matrix elements of the four-diagonal left-hand-side matrix are just linear functions of parameters  $a$  and  $b$ ,

$$\mathcal{M}_{k,k+2} = (k+1)(k+2), \quad k = 0, 1, \dots, N-2, \quad (27)$$

$$\mathcal{M}_{m,m+1} = 2b(m+1), \quad \mathcal{M}_{m+1,m} = -2(N-m), \quad m = 0, 1, \dots, N-1 \quad (28)$$

and

$$\mathcal{M}_{n,n} = 4an + 2a + b^2, \quad n = 0, 1, \dots, N. \quad (29)$$

Whenever real, the  $j$ -th eigenvalue may then represent an auxiliary energy parameter,

$$E = -p = -p_j(a, b, N). \quad (30)$$

Naturally, in the light of the illustrative  $N = 2$  example of preceding paragraph the parameters  $a$  and  $b$  must guarantee the wave-function matching in the origin. Thus, their values must be determined as roots of the constraints, i.e., as roots of the coupled pair of the above-discussed additional nonlinear equations

$$v_0^{(even)}(a, b, N, j) = 1, \quad v_1^{(even)}(a, b, N, j) = -b \quad (31)$$

(for the arbitrarily normalized even-parity QES states) or

$$v_0^{(odd)}(a, b, N, j) = 0, \quad v_1^{(odd)}(a, b, N, j) = 1 \quad (32)$$

(for the arbitrarily normalized odd-parity QES states).

## 4 Summary

In summary, the use of branched asymptotics (7) + (12) and of the related non-analytic version (8) of ansatz (5) was shown here to lead to the construction of an entirely new family of unconventional, asymptotically quartic QES oscillators (3). They were shown to share a number of useful closed-form-solvability properties with their conventional, everywhere analytic sextic-oscillator predecessors. In other words, we managed to prove that the non-analytic but asymptotically quartic potential well (3) may be perceived as quasi-exactly solvable. Constructively we demonstrated that the potential offers a bound-state model which is exactly, non-numerically solvable in terms of polynomials in  $x$  at certain couplings  $A$ ,  $B$ ,  $C$  and energies  $E_n$ .

The main technical ingredient which made the traditional QES recipe perceivably (and also, for our purposes, *sufficiently*) more flexible may be seen in our deliberate violation of the conventional analyticity assumption at a single point, viz., at  $x = 0$ . This opened the possibility of making the potential (as well as the whole Hamiltonian) spatially symmetric (i.e.,  $\mathcal{P}$ -symmetric), with the well known consequence of having also all of the bound states  $\psi_n(x)$  (and, in particular, also the exceptional QES states) characterized by their even or odd parity,  $\mathcal{P}\psi_n(x) = \psi_n(-x) = \pm\psi_n(x)$ .

The rest of our present story just forms a new, innovative but still rather close parallel to the conventional (or, equally well, to the above-mentioned less conventional, manifestly non-Hermitian) linear-algebraic QES constructions as described in an extensive dedicated literature (we may recommend the monograph [1] as a source of further references).

Naturally, in the future the implementation of the idea need not remain restricted to the present, piecewise analytic quartic interaction example. Still, we believe that due to an exceptional methodical as well as practical (e.g., computation-testing) role of the quartic-interaction class of models (with a single-point non-analyticity in our present case) might lead to their quick inclusion in the currently existing list of the available QES quantum systems, with all of their valuable practical applications as thoroughly reviewed, e.g., by Ushveridze [1].

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## Appendix A. A few comments on terminology

In the literature the specification of the concept of the exact solvability (ES) of Schrödinger equations is often vague and formulated *ad hoc*. For example, many people exclusively assign the exceptional ES status to the simplest square-well models in which the motion of a confined particle remains, locally, a free motion. Another community of physicists admits solely analytic forms of ES potentials  $V(x)$  requiring, in addition, that the ordinary differential bound-state Schrödinger Eq. (2) remains solvable in terms of classical orthogonal polynomials [16].

Whichever definition one accepts, the ES models remain separated from the generic, purely numerical ones by an equally vaguely specified grey zone in which, typically, one generalizes the piecewise constant potentials  $V(x)$  and takes advantage of a move to the mere distributional, delta-function-like point interactions [17]. Alternatively, the separated, analytic-function community perceives the grey solvability zone as covering, say, the transition from the three-dimensional Coulomb potential  $V^{(CO)}(\vec{r}) = -e^2/|\vec{r}|$  (which is solvable in terms of Laguerre polynomials) to its Ishkhanyan's [18] long-range ES modification  $V^{(Ish)}(\vec{r}) = -e^2/\sqrt{|\vec{r}|}$  in  $s$ -wave. The difference is that the latter model only proves solvable in terms of *non-terminating* confluent hypergeometric functions. For this reason, the bound-state energies themselves still do have just a numerical, non-ES, grey-zone (GZ) status.

Via an elementary change of variables the ES status of  $V^{(CO)}(\vec{r})$  (living on half-line) is shared with the quadratic harmonic-oscillator polynomial interaction  $V^{(HO)}(x) = Ax + x^2$  living on the whole real line of Eq. (2). Similarly, the semi-numerical, GZ solvability status of  $V^{(Ish)}(\vec{r})$  is formally shared with the analytic quartic oscillator (1) on the line. This form of correspondence reflects the reducibility of Eq. (2) + (1) to the so called Heun's differential equation [18] which is just “next” to the hypergeometric family and which still possesses a number of exceptional GZ features [19].

Inside such a GZ classification pattern the position of the traditional QES class is fully inside the analytic-potential area. For illustration people usually recall the formally privileged status of the sextic model (4) while emphasizing that, formally speaking, the model may be interpreted as an immediate successor of harmonic oscillator as well as an immediate predecessor of quartic oscillator of Eq. (1). The partial solvability of the even-parity sextic oscillator makes it formally privileged in comparison with the spatially asymmetric quartic polynomial (1) containing one more dynamics-determining coupling constant.

Still, the conventional characterization of the three-parametric quartic oscillators as “purely numerical” is not entirely deserved, for several reasons. The main one has already been mentioned above: The underlying ordinary differential Schrödinger equation belongs to the special class of Heun equations [19]. Among the multiple benefits of using these next-to-hypergeometric GZ equations we already mentioned the recent discovery [18] of the exact  $s$ -wave solvability of bound states in the long-range central potential  $V(\vec{r}) \sim 1/\sqrt{|\vec{r}|}$ . Marginally, let us now add that the phenomenological friendliness makes the quartic oscillators and potential functions Eq. (1) rather

popular even out of quantum mechanics. Typically, they found applications even in the theory of classical dynamical systems where the so called Lyapunov function of the functional form (1) can simulate one of the most widespread bifurcation-evolution scenarios called “cusp catastrophe” [20, 21, 22]).

In 1998, an apparently impenetrable formal boundary between the domains of quartic oscillators and of QES oscillators was broken by Bender and Boettcher [7]. They admitted the purely numerical status of potentials (1) but they discovered a way out of the trap. In brief, they demonstrated, constructively, that the elementary analytic solvability of the conventional (i.e., Hermitian) sextic model (4) *survives* the transition to certain modified, non-conventional quartic-oscillator Hamiltonians. The core of their proposal lied in the replacement of the real and confining potential (1) by its complex plus asymptotically “wrong-sign” alternative

$$V(x) = V^{(BB)}(x) = iAx + Bx^2 + iCx^3 - x^4. \quad (33)$$

Although such a generalization already lies far beyond the scope of our present paper (cf. also a few related comments in [9, 11, 12]), we only have to emphasize, in the conclusion, that measured by the degree of solvability, analytic model (4) and non-analytic model (33) certainly belong to the same (viz., QES) category, irrespectively of the subtle details of its scope and rigorous definition.